Referance answer of Lecture 5

Example 5.2

Fine the bound of solution for the IVP $x' = f(t,x) = -(1+x^2)x + e^t$, $x(0) = a$ without solving the equation.

Proof:Let

$v(t) = x^2(t)$,then $v' = 2xx' = 2x(-x - x^3 + e^t) = -2x^2 - 2x^4 + 2xe^t \le 2xe^t \le x^2 + e^{2t}$

$$\Rightarrow \begin{cases} v' \le v + e^{2t} \\ v(0) = a^2 \end{cases}$$

Let $\begin{cases} u' = u + e^{2t} \\ u(0) = a^2 \end{cases} \Rightarrow u = (e^t - 1 + a^2)e^t, t \in [0, t_1] \subset [0, \beta)$

$$\Rightarrow v \le u \le (e^\beta - 1 + a^2)e^\beta < \infty \Rightarrow x = \sqrt{v} < \infty \quad \forall t \in [0, t_1] \subset [0, \beta)$$

From the continuation theorem we know that $x = \sqrt{v} < \infty \quad \forall t \ge 0$

Remark5.8 You may prove Theorem 5.5 with $I_{\max} = (-\infty, +\infty)$ by Gronwall's inequality.

Proof:If $I_{\max} = [t_0, \omega_+], \omega_+ < +\infty$

Then from $x' = A(t)x + h(t) \Rightarrow x(t) = x(t_0) + \int_{t_0}^{t} [A(s)x(s) + h(s)]ds$

$$\begin{aligned} \Rightarrow \|x(t)\| &\le \|x(t_0)\| + \int_{t_0}^{t} \|A(s)x(s) + h(s)\| ds \\ &\le \|x(t_0)\| + \int_{t_0}^{t} \|h(s)\| ds + \int_{t_0}^{t} \|A(s)x(s)\| ds \\ &\le \|x(t_0)\| + \int_{t_0}^{\omega_+} \|h(s)\| ds + \int_{t_0}^{t} \|A(s)x(s)\| ds \end{aligned}$$

Then use the Gronwall's inequality we get

$$\|x(t)\| \le \{\|x(t_0)\| + \int_{t_0}^{\omega_+} \|h(s)\| ds\} e^{\int_{t_0}^{t} \|A(s)\| ds} \le \{\|x(t_0)\| + \int_{t_0}^{\omega_+} \|h(s)\| ds\} e^{\int_{t_0}^{\omega_+} \|A(s)\| ds} < +\infty$$

Contradiction.

So we have $\omega_+ = +\infty$.

Similarly we get $\omega_- = -\infty$.

This completes the proof.